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Class model of the complex fuzzy spatial object

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Contents

Summary

This report presents the class model of the complex fuzzy spatial object. Atlas Platform class model of the complex spatial object relies upon two basic models:

- Linear fuzzy space proposed in [1], and
- Formal object model of the geographic data proposed in [2].

Linear fuzzy space mathematical models provide for basic planar imprecise geometric objects (fuzzy line, fuzzy polygon, fuzzy triangle, fuzzy circle), as well as the basic measurement functions (distance between fuzzy sets representing basic planar imprecise object), fuzzy spatial operations (fuzzy union, fuzzy intersection and convex fuzzy hull), and fuzzy spatial relations (coincidence, between and collinear). Then, the class model of the complex fuzzy spatial object is obtained by applying linear fuzzy space mathematical models to a formal object model of geographic data.

Keywords

Fuzzy point, linear fuzzy space, fuzzy line, fuzzy triangle, fuzzy collinear, spatial object, Fuzzy Spatial Region, Fuzzy Spatial Field, Fuzzy Spatial Object, and Fuzzy Spatial Object Map

1. Introduction

Modelling imprecision and vagueness in spatially determined systems is an issue that attracts wide attention of numerous researchers in diverse application domains. Environmental science is not an exemption to that rule. On the contrary, the data in this field are often imprecise (geographical location, time, observed/measured values, etc). On the other hand, a huge demand for modelling complex spatial objects in environmental sciences calls for appropriate models that include uncertainty, imprecision, and vagueness.

In this report we present a class model of the complex planar imprecise object in ATLAS platform.

The rest of this report consists of four sections. Following this introduction is a section two that presents fundamentals of the simple spatial objects modelling. Then, the section three presents formal object model of the geographic data proposed in [2]. Section four presents the proposed model of the complex planar imprecise object in ATLAS platform. Section five is a conclusion that discusses advantages and disadvantages of the proposed model.

2.1. Simple object modelling

2.1.1.Fuzzy point, linear fuzzy space, fuzzy relation

Definition 2.1.1 *Fuzzy point* $P \in \mathbb{R}^2$, denoted by $\tilde{P}$ is defined by its membership function $\mu_{\tilde{P}} \in \mathcal{F}^2$**,** where the set $\mathcal{F}^2$ contains all membership functions $u: \mathbb{R}^2 \to [0,1]$ satisfying following conditions:

i) $(\forall u \in \mathcal{F}^2)(\exists_1 P \in \mathbb{R}^2)\ u(P) = 1$,
ii) $(\forall X_1, X_2 \in \mathbb{R}^2)(\lambda \in [0,1])\ u(\lambda X_1 + (1-\lambda)X_2) \geq \min\big(u(X_1), u(X_2)\big)$,
iii) function u is upper semi continuous,
iv) $[u]^\alpha = \{X | X \in \mathbb{R}^2, u(X) \geq \alpha\}$ α-cut of function u is convex.

The point from $\mathbb{R}^2$, with membership function $\mu_{\tilde{P}}(P) = 1$, will be denoted by P (P is the core of the fuzzy point $\tilde{P}$), and the membership function of the point $\tilde{P}$ will be denoted by $\mu_{\tilde{P}}$. By $[P]^\alpha$ we denote the α-cut of the fuzzy point (this is a set from $\mathbb{R}^2$).

Definition 2.1.2 $\mathbb{R}^2$ *Linear fuzzy space* is the set $\mathcal{H}^2 \subset \mathcal{F}^2$ of all functions which, in addition to the properties given in Definition 2.1, are:

i) Symmetric against the core $S \in \mathbb{R}^2$
$(\mu(S) = 1)$,
$\mu(V) = \mu(M) \wedge \mu(M) \neq 0 \Rightarrow d(S,V) = d(S,M)$,
where $d(S,M)$ is the distance in $\mathbb{R}^2$.
ii) Inverse-linear decreasing w.r.t. points' distance from the core according to:
If $r \neq 0$
$$\mu_{\tilde{S}}(V) = \max\left(0, 1 - \frac{d(S,V)}{|r_S|}\right),$$
if $r = 0$
$$\mu_{\tilde{S}}(V) = \begin{cases} 1 & if \quad S = V \\ 0 & if \quad S \neq V, \end{cases}$$
where $d(S,V)$ is the distance between the point V and the core S ($V, S \in R^n$) and $r \in \mathbb{R}$ is constant.

Elements of that space are represented as ordered pairs $\tilde{S} = (S, r_S)$ where $S \in \mathbb{R}^2$ is the core of $\tilde{S}$, and $r_S \in \mathbb{R}$ is the distance from the core for which the function value becomes 0; in the sequel parameter r_S will be denoted as *fuzzy support radius*.

Definition 2.1.3 Let the linear fuzzy space $\mathcal{H}$ be defined on $\mathbb{R}$. Fuzzy relations $\leq^{RF}$ and $\leq^{LF}$ for the set $\mathcal{H}$ are defined by membership functions

$$\mu(\tilde{A} \leq^{RF} \tilde{B}) = \begin{cases} 0 & if \;\; A > B, \\ \frac{B-A}{r_A - r_B} & if \;\; A \leq B \wedge A + r_A > B + r_B \\ 1 & if \;\; A \leq B \wedge A + r_A \leq B + r_B \,, \end{cases}$$

$$\mu(\tilde{A} \leq^{LF} \tilde{B}) = \begin{cases} 0 & if \;\; A > B \\ \dfrac{B-A}{r_B - r_A} & if \;\; A \leq B \wedge \; A - r_A > B - r_B \\ 1 & if \;\; A \leq B \wedge A - r_A \leq B - r_B, \end{cases}$$

respectively, where $\tilde{A} = (A, r_A)$ and $\tilde{B} = (B, r_B)$ are points from $\mathcal{H}$, A is the core of $\tilde{A}$ and r_A is a parameter determining the membership function of point $\tilde{A}$.

2.1.2.Basic fuzzy plane geometry objects in R^2 linear fuzzy space

In this section we present theoretical models of basic operations over linear fuzzy space $\mathcal{H}^2$ defined on $\mathbb{R}^2$, as well as their properties which will be used in definitions of basic fuzzy plane geometry objects.

Definition 2.2.1 Let $\tilde{A}, \tilde{B} \in \mathcal{H}^2$. An operator $+ : \mathcal{H}^2 \times \mathcal{H}^2 \to \mathcal{H}^2$ is called *fuzzy points addition* given by

$$\tilde{A} + \tilde{B} = (A + B, r_A + r_B),$$

where $A + B$ is a vector addition, and $r_A + r_B$ is a scalar addition.

Definition 2.2.2 Let $\mathcal{H}^2$ be a linear fuzzy space. Then a function $f : \mathcal{H}^2 \times \mathcal{H}^2 \times [0,1] \to \mathcal{H}^2$ is called *linear combination* of the fuzzy points $\tilde{A}, \tilde{B} \in \mathcal{H}^2$ given by

$$f(\tilde{A}, \tilde{B}, u) = \tilde{A} + u \cdot (\tilde{B} - \tilde{A}),$$

where $u \in [0,1]$ and operator $\cdot$ is a scalar multiplication of fuzzy point.

Definition 2.2.3 Let $\tilde{A}, \tilde{B} \in \mathcal{H}^2$ and $\tilde{A} \neq \tilde{B}$. Then a point $T_{AB} \in \mathbb{R}^2$ is called *internal homothetic center* if the following holds

$$T_{AB} = A + \frac{a_r}{a_r + b_r}(B - A),$$

where $\tilde{A} = (A, a_r)$ and $\tilde{B} = (B, b_r)$.

Fuzzy points are used to describe the position of a real object when there is some uncertainty to the measured position. Most often this uncertainty in practical applications is ignored. There are applications in which real objects are not only represented by the position but the entire series of uniformly spaced points. These points can be distributed along a curve that has a beginning and an end. Curve that connects two points is called a line or path.

If the points that represent the path are imprecise, then the whole line should be described in way similar to imprecise point's description. The mathematical model of such fuzzy line follows.

Definition 2.2.4 Let $\mathcal{H}^2$ be a linear fuzzy space and function f is a linear combination of the fuzzy points $\tilde{A}$ and $\tilde{B}$. Then a fuzzy set $\widetilde{AB}$ is called *fuzzy line* if following holds

$$\widetilde{AB} = \bigcup_{u\in[0,1]} f(\tilde{A},\tilde{B},u).$$

Theorem 2.2.1 Let $\mathcal{H}^2$ be linear fuzzy space, fuzzy line $\widetilde{AB}$ defined by fuzzy points $\tilde{A}$ and $\tilde{B} \in \mathcal{H}^2$. Then following holds

$\widetilde{AB} = \widetilde{BA}$.

Definition 2.2.5 Let $\widetilde{AB}$ be *fuzzy line* defined on linear fuzzy space $\mathcal{H}^2$ and $X \in \mathbb{R}^2$. Then a fuzzy point $\widetilde{X'} \subset \widetilde{AB}$ is called *fuzzy image of point X on fuzzy line $\widetilde{AB}$*, and a real number $u \in [0,1]$ is called *eigenvalue of the fuzzy image X* on fuzzy line $\widetilde{AB}$ if following hold

(i) $\widetilde{X'} = \tilde{A} + u(\tilde{B} - \tilde{A})$,
(ii) $d\left(X, \left[\widetilde{X'}\right]^1\right) = \min\{d(X,Y) | \forall Y \in \left[\widetilde{AB}\right]^1\}$,
(iii) $u = \min\left(1, max\left(0, \frac{(x_1-a_1)(b_1-a_1)+(x_2-a_2)(b_2-a_2)}{(b_1-a_1)^2+(b_2-a_2)^2}\right)\right)$,

where $X = (x_1, x_2)$, $\tilde{A} = ((a_1,a_2),a_r)$ i $\tilde{B} = ((b_1,b_2),b_r)$.

Remark. If the eigenvalue of the fuzzy image X is equal 0, then fuzzy image is the starting fuzzy point, if eigenvalue is equal 1 it is the final point, otherwise it is the inner point of a fuzzy line.

Theorem 2.2.2 Let $\widetilde{AB} \in L^2$ be fuzzy line, $\widetilde{X'} \in \mathcal{H}^2$ fuzzy image of point $X \in \mathbb{R}^2$ on $\widetilde{AB}$ and $u \in [0,1]$ eigenvalue of the fuzzy image X on $\widetilde{AB}$. Then point X belongs to fuzzy set $\widetilde{AB}$ according to following

$$\mu_{\widetilde{AB}}(X) = \begin{cases} \mu_{\tilde{A}}(X) & if \quad u_m = 0 \\ \mu_{\tilde{X}_T}(X) & if\ 0 < u_m < 1 \\ \mu_{\tilde{B}}(X) & if \quad u_m = 1 \end{cases},$$

where fuzzy point $\tilde{X}_T = \tilde{A} + u_m(\tilde{B} - \tilde{A})$ and $u_m = u + \frac{(b_r - a_r)}{x'_r}\frac{d(X,X')^2}{d(A,B)^2}$.

Definition 2.2.6 Let $\tilde{A}$, $\tilde{B}$, $\tilde{C} \in \mathcal{H}^2$ be fuzzy points with noncollinear cores ($\tilde{A} \neq \tilde{B} \neq \tilde{C}$) and function f is a linear combination of two fuzzy points. Then the fuzzy set $\widetilde{ABC}$ is called a *fuzzy triangle* if the following holds

$$\widetilde{ABC} = \bigcup_{u=0}^{1} f(\tilde{A}, \bigcup_{v=0}^{1} f(\tilde{B},\tilde{C},v),u)$$

The membership function of this set is denoted by $\mu_{\widetilde{ABC}}(X)$ and determined according to the following formula

$$\mu_{\widetilde{ABC}}(X) = \max_{u\in[0,1],v\in[0,1]}\{\mu_{\tilde{Y}}(X) |\ \tilde{Y} = f(\tilde{A}, f(\tilde{B},\tilde{C},v),u)\}.$$

α-cut of fuzzy triangle $\widetilde{ABC}$ is denoted by $\left[\widetilde{ABC}\right]^\alpha$.

Definition 2.2.7 Let $\widetilde{ABC}$ be a fuzzy triangle defined on fuzzy linear space $\mathcal{H}^2$. Fuzzy point $\tilde{X} \subset \widetilde{ABC}$ is called *edge point of the fuzzy triangle $\widetilde{ABC}$* if for all $\alpha \in [0,1]$ a point $Y \in [\tilde{X}]^\alpha$ exists such that all its neighborhoods contain at least one point from $\left[\widetilde{ABC}\right]^\alpha$ and at least one point outside of $\left[\widetilde{ABC}\right]^\alpha$.

Remark. α-cut of all edge points intersect α-cut of fuzzy triangle in at least one point.

Definition 2.2.8 Let $\widetilde{ABC}$ be a fuzzy triangle defined on fuzzy linear space $\mathcal{H}^2$. Fuzzy point $\tilde{X} \subset \widetilde{ABC}$ is called *inner point of fuzzy triangle* $\widetilde{ABC}$ if it is not an edge point.

Definition 2.2.9 Let $\widetilde{ABC}$ be a fuzzy triangle defined on fuzzy linear space $\mathcal{H}^2$. Union of all edge points of the fuzzy triangle $\widetilde{ABC}$ is called fuzzy edge of fuzzy triangle $\widetilde{ABC}$, denoted by $\partial\widetilde{ABC}$.

Theorem 2.2.3 Let $\widetilde{ABC}$ be a fuzzy triangle defined on $\mathcal{H}^2$. Then, for every fuzzy point $\tilde{X} \in \partial\widetilde{ABC}/\{\tilde{A}, \tilde{B}, \tilde{C}\}$ and $\alpha \in [0,1]$ the single point $T \in [\tilde{X}]^\alpha$ exists such that all its neighborhoods contain at least one point from $\left[\widetilde{ABC}\right]^\alpha$ and at least one point outside of $\left[\widetilde{ABC}\right]^\alpha$.

Theorem 2.2.4 Let $\widetilde{ABC}$ be a fuzzy triangle defined on linear fuzzy space $\mathcal{H}^2$. Then for all $X \in \mathbb{R}^2$ the following holds

$$\mu_{\widetilde{ABC}}(X) = \mu_{\widetilde{CAB}}(X) = \mu_{\widetilde{BCA}}(X).$$

Direct consequence of this proposition is that a fuzzy triangle can be represented by three fuzzy points, i.e., the set $\{\tilde{A}, \tilde{B}, \tilde{C}\}$.

Fuzzy circle is also one of the basic planar imprecise geometrical objects. Analogously to the definitions of fuzzy line and fuzzy triangle, which is an extension of a precise circle, we define a fuzzy circle as a union of fuzzy points. Thereby, we also take care that a newly defined geometrical object is appropriate for implementation in GIS applications.

Definition 2.2.10 Let $\mathcal{H}$ be a fuzzy space defined on $\mathbb{R}$, fuzzy relation $\leq^{RF}$ be fuzzy ordering in linear fuzzy space $\mathcal{H}$, $C \in \mathbb{R}^2$ and $\tilde{R} \in \mathcal{H}$. Then the union of all fuzzy points $\tilde{A} \in \mathcal{H}^2$ such that

$$\mu\left(\tilde{d}\left(C, \tilde{A}\right) \leq^{RF} \tilde{R}\right) = 1,$$

is called *fuzzy circle* with center C and radius $\tilde{R}$.
Fuzzy circle is represented by the ordered pair $(C, \tilde{R})$.

Theorem 2.2.5 Let $\left(C, \tilde{R}\right)$ be a fuzzy circle defined on linear fuzzy space $\mathcal{H}^2$. Then the value of the fuzzy circle membership function in point $X \in R^2$ is determined according to the following formula

$$\mu_{(C,\tilde{R})}(X) = \max\left(0, min\left(1, 1 - \frac{d(X,C) - R}{r_r}\right)\right),$$

where $\tilde{R} = (R, r_r)$.

Definition 2.2.11. Let $\mathcal{H}^2$ be a linear fuzzy space and $\tilde{\mathcal{A}} = \{\widetilde{A_1}, .. \widetilde{A_n}\}$ be the ordered set of the fuzzy points $\widetilde{A_\iota} \in \mathcal{H}^2$. Then *linear fuzzy path* $s(\tilde{\mathcal{A}})$ is given by

$$s(\tilde{\mathcal{A}}) = \bigcup_{i=1}^{n-1} \widetilde{A_\iota A_{\iota+1}}$$

If $X \in \mathbb{R}^2$, then the membership function $\mu_{s(\tilde{\mathcal{A}})}$ of the linear fuzzy path $s(\tilde{\mathcal{A}})$ is given by

$$\mu_{s(\tilde{\mathcal{A}})}(X) = \max_{i=1,n-1} \mu_{\widetilde{A_\iota A_{\iota+1}}}(X).$$

Definition 2.2.12 Let $\mathcal{H}^2$ be a linear fuzzy space and $\tilde{\mathcal{A}} = \{\widetilde{A_1}, .. \widetilde{A_n}\}$ be the ordered set of the fuzzy points $\widetilde{A_\iota} \in \mathcal{H}^2$.Then *closed linear fuzzy path* $c(\tilde{\mathcal{A}})$ is given by

$$c(\tilde{\mathcal{A}}) = s(\tilde{\mathcal{A}}) \bigcup \widetilde{A_n A_1}$$

If $X \in \mathbb{R}^2$, then the membership function $\mu_{c(\tilde{\mathcal{A}})}$ of the closed linear fuzzy path $c(\tilde{\mathcal{A}})$ is given by

$$\mu_{c(\tilde{\mathcal{A}})}(X) = \max\{\mu_{s(\tilde{\mathcal{A}})}(X), \mu_{\widetilde{A_n A_1}}(X)\}.$$

Definition 2.2.13 Let $\mathcal{H}^2$ be a linear fuzzy space and $\tilde{\mathcal{A}} = \{\widetilde{A_1}, .. \widetilde{A_n}\}$ be the ordered set of the fuzzy points $\widetilde{A_\iota} \in \mathcal{H}^2$. Then, a *linear fuzzy polygon* $p(\tilde{\mathcal{A}})$ is given by

$$\mu_{p(\tilde{\mathcal{A}})}(X) = \begin{cases} 1 & if\ X\ inside\ polygon\ c(\tilde{\mathcal{A}})^0 \\ \mu_{c(\tilde{\mathcal{A}})}(X) & otherwise\ , \end{cases}$$

where $c(\tilde{\mathcal{A}})^0$ is the core of the fuzzy set $c(\tilde{\mathcal{A}})$.

2.1.3. Spatial measurement in $\mathbf{R^2}$ linear fuzzy space

Measurement of the space, especially a distance between plane geometry objects is defined as a generalization of the concept of physical distance. Distance function or metric is a function that behaves according to specific set of rules. In this section we present the basic distance functions between fuzzy plane geometry objects and their main properties according to the set of rules presented in papers [1], [20]

Definition 2.3.1 Let $\mathcal{H}^2$ be a linear fuzzy space, $\tilde{d}: \mathcal{H}^2 \times \mathcal{H}^2 \to \mathcal{H}^+$, $L, R: [0,1] \times [0,1] \to [0,1]$ be symmetric, associative and non-decreasing for both arguments, and $L(0,0) = 0$, $R(1,1) = 1$. The ordered quadruple $(\mathcal{H}^2, \tilde{d}, L, R)$ is called fuzzy metric space and the function $\tilde{d}$ is a *fuzzy metric*, if and only if the following conditions hold:

(i) $\tilde{d}(\tilde{X}, \tilde{Y}) = \tilde{0} \iff [\tilde{X}]^1 = [\tilde{Y}]^1$.

(ii) $\tilde{d}(\tilde{X}, \tilde{Y}) = \tilde{d}(\tilde{Y}, \tilde{X})$ for each $\tilde{X}, \tilde{Y} \in \mathcal{H}^2$.

(iii) $\forall \tilde{X}, \tilde{Y} \in \mathcal{H}^2$:

a. $\tilde{d}(\tilde{X}, \tilde{Y})(s+t) \geq L\big(d(x,z)(s), d(z,y)(t)\big)$ if
$s \leq \lambda_1(x,z) \wedge\ t \leq \lambda_1(z,y) \wedge\ s+t \leq\ \lambda_1(x,y)$

b. $\tilde{d}(\tilde{X}, \tilde{Y})(s+t) \leq R\big(d(x,z)(s), d(z,y)(t)\big)$ if
$s \geq \lambda_1(x,z) \wedge\ t \geq \lambda_1(z,y) \wedge\ s+t \geq\ \lambda_1(x,y)$,

where the α-cut of fuzzy number $\tilde{d}(x,y)$ is given by $\big[\tilde{d}(\tilde{X}, \tilde{Y})\big]^\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)]$ $(x, y \in \mathbb{R}^+, 0 < \alpha \leq 1)$. The fuzzy zero, $\tilde{0}$ is a *non-negative* fuzzy number with $[\tilde{0}]^1 = 0$.

Remark: Following distance functions are fuzzy metrics.

(i) $\tilde{d}(\tilde{X}, \tilde{Y}) =_{DF} (d(X,Y), (r_X + r_Y))$

(ii) $\tilde{d}(\tilde{X}, \tilde{Y}) =_{DF} (d(X,Y), max(r_X, r_Y))$

(iii) $\tilde{d}(\tilde{X}, \tilde{Y}) =_{DF} (d(X,Y),\ |r_X - r_Y|)$

Distance (iii) also satisfies set of rulles which define clasic metric.

In the following definitions we extend distance between fuzzy points to distance between different fuzzy plane geometric objects, such as distance between fuzzy point and fuzzy line, fuzzy point and fuzzy triangle and at last between two fuzzy triangles.

Definition 2.3.2 Let $\mathcal{H}^2$ be a linear fuzzy space, $\mathcal{L}^2$ set of all fuzzy linesdefined on $\mathcal{H}^2$, $\tilde{d}$ is fuzzy distance between fuzzy points, and μ_L is membership function of the fuzzy relation minimal (Definition 15. in the paper [1]). The function $dist: \mathcal{H}^2 \times \mathcal{L}^2 \rightarrow \mathcal{H}^+$ is called *distance between fuzzy point and fuzzy line* if the following holds:

$$dist\left(\tilde{T}, \widetilde{AB}\right) = \tilde{d}(\tilde{T}, \tilde{X})$$

where $\tilde{X} \in \widetilde{AB}$ such that $\mu_L\left(\tilde{d}\left(\tilde{T}, \tilde{X}\right)\right) = hgt\left(\left\{\tilde{d}\left(\tilde{T}, \tilde{Y}\right) \middle| \forall \tilde{Y} \in \widetilde{AB}\right\}\right)$.

Definition 2.3.3 Let $\mathcal{H}^2$ be linear fuzzy space, $\mathcal{T}^2$ be a set of all fuzzy triangles defined on $\mathcal{H}^2$, $\tilde{d}$ is fuzzy distance between fuzzy points and μ_L is membership function of the fuzzy relation minimal. The function $dist: \mathcal{H}^2 \times \mathcal{T}^2 \rightarrow \mathcal{H}^+$ is called distance between fuzzy point and fuzzy triangle if the following holds:

$$dist\left(\tilde{T}, \widetilde{ABC}\right) = \tilde{d}(\tilde{T}, \tilde{X})$$

where $\tilde{X} \in \widetilde{ABC}$ such that $\mu_L\left(\tilde{d}\left(\tilde{T}, \tilde{X}\right)\right) = hgt\left(\left\{\tilde{d}\left(\tilde{T}, \tilde{Y}\right) \middle| \forall \tilde{Y} \in \widetilde{ABC}\right\}\right)$

Definition 2.3.4 Let $\mathcal{H}^2$ be linear fuzzy space, $\mathcal{L}^2$ set of all fuzzy lines on $\mathcal{H}^2$ $\tilde{d}$ is fuzzy distance between fuzzy points and μ_L is membership function of the fuzzy relation minimal. Th function $dist: \mathcal{L}^2 \times \mathcal{L}^2 \rightarrow \mathcal{H}^+$ is called distance between two fuzzy lines if the following holds:

$$dist\left(\widetilde{AB}, \widetilde{CD}\right) = \tilde{d}(\tilde{X}, \tilde{Y})$$

where $\tilde{X} \in \widetilde{AB}$ and $\tilde{Y} \in \widetilde{CD}$ such that

$$\mu_L\left(\tilde{d}\left(\tilde{X}, \tilde{Y}\right)\right) = hgt\left(\left\{\tilde{d}\left(\tilde{Q}, \widetilde{W}\right) \middle| \forall \tilde{Q} \in \widetilde{AB} \land \forall \widetilde{W} \in \widetilde{CD}\right\}\right)$$

Definition 2.3.5 Let $\mathcal{H}^2$ be a linear fuzzy space, $\mathcal{L}^2$ be a set of all fuzzy lines on $\mathcal{H}^2$, $\mathcal{T}^2$ be a set of all fuzzy triangles, $\tilde{d}$ is fuzzy distance between fuzzy points and μ_L is membership function of the fuzzy relation minimal. The function $dist: \mathcal{L}^2 \times \mathcal{T}^2 \rightarrow \mathcal{H}^+$ is called distance between fuzzy line and fuzzy triangle if the following holds:

$$dist\left(\widetilde{AB}, \widetilde{CDE}\right) = \tilde{d}(\tilde{X}, \tilde{Y})$$

where $\tilde{X} \in \widetilde{AB}$ and $\tilde{Y} \in \widetilde{CDE}$ satisfies condition

$$\mu_L\left(\tilde{d}\left(\tilde{X}, \tilde{Y}\right)\right) = hgt\left(\left\{\tilde{d}\left(\tilde{Q}, \widetilde{W}\right) \middle| \forall \tilde{Q} \in \widetilde{AB} \land \forall \widetilde{W} \in \widetilde{CDE}\right\}\right)$$

Definition 2.3.6 Let $\mathcal{H}^2$ be linear fuzzy space, $\mathcal{T}^2$ be a set of all fuzzy triangles on $\mathcal{H}^2$, $\tilde{d}$ is fuzzy distance between fuzzy points and μ_L is membership function of the fuzzy relation minimal. The function $dist: \mathcal{T}^2 \times \mathcal{T}^2 \rightarrow \mathcal{H}^+$ is called distance between two fuzzy triangles if the following holds:

$$dist\left(\widetilde{ABC}, \widetilde{DEF}\right) = \tilde{d}(\tilde{X}, \tilde{Y})$$

where fuzzy points $\tilde{X} \in \widetilde{ABC}$ and $\tilde{Y} \in \widetilde{DEF}$ such that
$\mu_L\left(\tilde{d}\left(\tilde{X}, \tilde{Y}\right)\right) = hgt\left(\left\{\tilde{d}\left(\tilde{Q}, \widetilde{W}\right) \middle| \forall \tilde{Q} \in \widetilde{ABC} \land \forall \widetilde{W} \in \widetilde{DEF}\right\}\right)$.

2.1.4. Spatial relations in R^2 linear fuzzy space

Spatial relations (predicates) are functions that are used to establish mutual relations between the fuzzy geometric objects. The basic spatial relations are *coincide*, *between* and *collinear*. In this section we give their definitions and basic properties.
Fuzzy relation *coincidence* expresses the degree of truth that two fuzzy points are on the same place.

Definition 2.4.1 Let λ be the Lebesgue measure on the set [0,1] and $\mathcal{H}^2$ is a linear fuzzy space. The fuzzy relation $coin: \mathcal{H}^2 \times \mathcal{H}^2 \to [0,1]$ is *fuzzy coincidence* represented by the following membership function

$$\mu_{coin}(\tilde{A}, \tilde{B}) = \lambda(\{\alpha \mid [\tilde{A}]^\alpha \cap [\tilde{B}]^\alpha \neq \emptyset\}).$$

Remark. Since the lowest α is always 0, then a membership function of the *fuzzy coincidence* is given by

$$\mu_{coin}(\tilde{A}, \tilde{B}) = max\{\alpha \mid [\tilde{A}]^\alpha \cap [\tilde{B}]^\alpha \neq \emptyset\}.$$

Proposition
"*Fuzzy point $\tilde{A}$ is coincident to fuzzy point $\tilde{B}$*"
is partially true with the truth degree $\mu_{coin}(\tilde{A}, \tilde{B})$; in the Theorem 4.1 we present method for its calculation.

Theorem 2.4.1 Let the fuzzy relation *coin* be a fuzzy coincidence. Then the membership function of the fuzzy relation *fuzzy coincidence* is determined according to the following formula

$$\mu_{coin}(\tilde{A}, \tilde{B}) = \begin{cases} 0 & if \quad |a_r| + |b_r| = 0 \ \wedge d(A,B) \neq 0, \\ \max\left(0, 1 - \dfrac{d(A,B)}{|a_r| + |b_r|}\right) & if \quad |a_r| + |b_r| \neq 0, \\ 1 & if \quad |a_r| + |b_r| = 0 \ \wedge d(A,B) = 0\,. \end{cases}$$

Fuzzy relation *contains* or *between* is a measure that fuzzy point belongs to fuzzy line or fuzzy line contains fuzzy point.

Definition 2.4.2 Let λ be Lebesgue measure on the set [0,1], $\mathcal{H}^2$ linear fuzzy space and $\mathcal{L}^2$ be set of all fuzzy lines defined on $\mathcal{H}^2$. Then fuzzy relation $contain: \mathcal{H}^2 \times \mathcal{L}^2 \to [0,1]$ is *fuzzy contain* represented by following membership function

$$\mu_{contain}(\tilde{A}, \widetilde{BC}) = \lambda(\{\alpha \mid [\tilde{A}]^\alpha \cap [\widetilde{BC}]^\alpha \neq \emptyset\}).$$

Remark. Its membership function could be also represented as

$$\mu_{contain}(\tilde{A}, \widetilde{BC}) = \lambda(\{\alpha \mid \exists u \in [0,1] \wedge \exists X \in [\tilde{A}]^\alpha \wedge \exists Y, Z \in [\widetilde{BC}]^\alpha \wedge \ X = Y + u(Z - Y)\}).$$

Proposition
"*Fuzzy line $\widetilde{BC}$ contain fuzzy point $\tilde{A}$*"
is partially true with the truth degree $\mu_{contain}(\tilde{A}, \widetilde{BC})$; in the Theorem 4.2 we present method for its efficient calculation.

Theorem 2.4.2 Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{H}^2$ be fuzzy points defined on $\mathcal{H}^2$ linear fuzzy space, $u \in [0,1]$ and $\widetilde{A'}$ be *fuzzy image of point A on fuzzy line $\widetilde{BC}$*. Points T_{AB} and T_{AC} are *internal homothetic centre* fuzzy points for fuzzy points $\tilde{A}$ and $\tilde{B}$ and $\tilde{A}$ and $\tilde{C}$ respectively. Then the membership function of the fuzzy relation *fuzzy contain* is determined according to the following formula

$$\mu_{contain}(\tilde{A}, \widetilde{BC}) = \begin{cases} \mu_{coin}(\tilde{A}, \widetilde{A'})\ ako\ je\ u \in \{0,1\} \\ \mu_{\tilde{A}}(A^*) ako\ je\ u \in (0,1) \end{cases},$$

where point A^* is a projection of core of $\tilde{A}$ on the line passing through the points T_{AB} and T_{AC}.

Collinearity is also one of the fundamental relations between three points in plane geometry. In the following definition we will present our definition of *fuzzy collinearity* in fuzzy linear space, as well as the method for its practical computation.

Definition 2.4.3 Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{H}^2$ be a fuzzy points defined on $\mathcal{H}^2$ linear fuzzy space and λ be Lebesgue measure on the set [0,1]. The fuzzy relation $coli: \mathcal{H}^2 \times \mathcal{H}^2 \times \mathcal{H}^2 \to [0,1]$ is *fuzzy collinearity* between three fuzzy points and it is represented by following membership function

$$\mu_{coli}(\tilde{A}, \tilde{B}, \tilde{C}) = \lambda\{\alpha | \exists u \in R \wedge \exists X \in [\tilde{A}]^\alpha \wedge \exists Y \in [\tilde{B}]^\alpha \wedge \exists Z \in [\tilde{C}]^\alpha \wedge\ A = B + u(C - B)\}.$$

Proposition
"Fuzzy points $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are collinear"

is partially true with the truth degree $\mu_{coli}(\tilde{A}, \tilde{B}, \tilde{C})$; in the Theorem 4.3 we present method for its calculation.

Theorem 2.4.3 Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{H}^2$, fuzzy relation $contain$ be *fuzzy contain*. Then a membership function of the fuzzy relation *fuzzy colinearity* is determined according to the following formula

$$\mu_{coli}(\tilde{A}, \tilde{B}, \tilde{C}) = \max\left(\mu_{contain}(\tilde{A}, \widetilde{BC}), \mu_{contain}(\tilde{B}, \widetilde{AC}), \mu_{contain}(\tilde{C}, \widetilde{AB})\right).$$

3. Formal object model of the geographic data

In this section we present the formal object model of the geographic data [2] which is the basis for our model of complex fuzzy objects presented in section 4.

Definition 3.1. An object *o* can be constructed out of other objects o_1, ...o_n, in which case *o* is called *complex* and o_1, ...o_n are called the *components* of *o*. If an object is not complex, then it is called *simple*. Classes can be structured into hierarchies; the ancestors of a class C in the hierarchy are called the *superclasses* of C. Classes are divided into *conventional classes* and *geographical classes* (or *geo-classes*). The geo-classes model geographical fields and objects, whereas the conventional classes correspond to classes whose instances are non-spatial objects. Each geographical class of objects has both locational and conventional attributes.

Definition 3.2. The *conventional* attributes are assumed to be derived from a universe ***U*** of *descriptive* attributes $\{A_1, \ldots, A_n\}$, defined on domains $D(A_1), \ldots, D(A_n)$.

3.1. Basic model hierarchy

The basic model hierarchy recognizes following classes: Graphical Region, Graphical Field, Geo-Object, and Geo-Object Map.

Definition 3.1.1 A set of points R which is a subset of $\Re^2$ is called a **geographical region**.

Definition 3.1.2 Let R be a geographical region. A **geo-field** f is an object $[a_1, \ldots a_n,\ \lambda\ ]$, where $a_i \in D(Ai)$ and $\lambda:\ R\ \rightarrow\ V$ defines a mapping between points in R and values on a domain V.
In this model, the geographical fields can be specialized. Depending on the range of the variable, the following subclasses of GEO-FIELD are defined:

- THEMATICAL - an instance of this class, called a *thematical geo-field*, defines a mapping $\lambda: R\ \rightarrow\ V$ such that V is a finite denumerable set. The elements of V are called *geo-classes* and, intuitively, define the *themes* of a thematical map.
- NUMERICAL - an instance of this class, called a *digital terrain model* or simply a DTM, defines a mapping $\lambda: R\ \rightarrow\ V$ such that V is the set of real values.
- REMOTESENSINGDATA - a specialization of the NUMERICAL class, whose instances have a range V which is a set of discrete values obtained by quantization of the response of the earth's surface to incident radiation, obtained by an active or passive sensor. This class is particularly useful to integrate remote sensing images into a GIS.

Geo-fields can be represented in a GIS in various formats; digital terrain models can be represented by regular grids or triangular grids, thematic maps can be represented by a topologically-structured set of vectors or by a symbolic array (raster representation), and images are usually represented by an array of values (raster representation).

Definition 3.1.3 Given a set of geographical regions $R_1, \ldots R_n$, a **geo-object** go is an object $[a_1, \ldots a_n, geo_1, \ldots, geo_n]$, composed by the values $a_i \in D(Ai)$ and by a set of geographical locations geo_i (where $geoi\ \subseteq\ R_i$). We shall indicate the i-th attribute of go by $go.A_i$ and the i-th geographical location of go by $go.R_i$.
Geo-objects represent individualizable entities of the geographic domain. They are phenomena that may have one or more *graphical representations*, which correspond to the geo-referenced set of coordinates that describe the object's location. In simple words, an object is a unique element that can be represented in one or more points in space, and which has various descriptive attributes. This definition allows for multiple geometrical representations to be assigned to the same geo-object.

Definition 3.1.4 Let R be a geographical region. A **geo-object map** mo is an object $[R, GO, geo]$ such that GO is a set of geo-objects and geo is a mapping $GO \rightarrow\ R$, which assigns, for each geo-object $go\ \in\ GO$, a location $geo(go)$ in R.

In a GIS, each geographical object is associated to one or more geographical locations. Since most applications do not deal with isolated elements in space, it is convenient to store the graphical representation of geo-objects together with its neighbors. These features lead to introduction of the concept of geo-object maps, which group together geo-objects for a given cartographic projection and geographical region.

3.2. Operations on geographical data

The very same model suggests three main types of geographical algebras, defining operations on geographical data:

- Fields algebra: manipulation of fields.
- Geo-objects algebra: descriptive and spatial properties based selection and query of geo-objects.
- Combined operations: generation of geo-object maps from fields, and generation of fields from geo-objects.

3.2.1. Fields algebra

There are three basic types of fields' operators in the model: **point**, **neighborhood** and **zonal**.

3.2.1.1. Point operators

A point operator produces a new geo-field, whose value in each point p depends only on the values in p in the input geo-fields. Point operations are specified as a mapping between the ranges of the input and output fields. Formal definition follows.

Definition 3.2.1 Let R be a geographical region, $V_1, V_2, .. V_{n+1}$ sets which define possible ranges for geo-fields, and F_i $(i = 1, .., n + 1)$ be the class of all geo-fields which have R as a location and V_i as its range.

The **point operation** $\Pi: F_1 \times F2 \rightarrow F_{n+1}$ induces a function p such that, for every geo-field $f_i \in F_i$ $(i = 1, .., n)$:

$f_{n+1}(p) = p\ (f_1(p), \ldots, f_n(p)), \forall\ p \in R$. where the spatial values of the output geo-field $f_{n+1} \in F_{n+1}$ are defined by the mapping $\lambda_{n+1}: M \rightarrow V_{n+1}$.

Point operators include transformation operators, mathematical functions, boolean operations, comparison operators and functions such as finding extremes and averages. The value of the output field at each location is a function only of the input values at the corresponding location.

3.2.1.2. Neighborhood operators

In this class of operators, the output field is computed based on the values of a continuously-varying surface in the neighborhood of each location of the input field. What follows is the formal definition of the neighborhood operations on geo-fields preceded by the definition of the concept neighborhood in geographical region.

Definition 3.2.2 Given a geographical region R, a set of $P \subseteq R$ is said to be *connected* iff, for any two points $p_1, p_2 \in P$ there is a line connecting these two points which is entirely contained in R. A ***neighborhood*** in R is a mapping $N: R \rightarrow 2^R$, such that $\forall\ p \in R,\ p \in N(p)$ and $N(p)$ are connected.

Definition 3.2.3 Let R be a geographical region and F_0 and F_1 the sets of geo-fields which are defined over R and whose range is $V_i, i = 0, 1$. Let $N: R \rightarrow 2^R$ and $\upsilon: 2^{V_1} \rightarrow V_0$. The neighborhood operation $\Psi: F_1 \rightarrow F_0$ *induced* by υ is such that:

$\forall f_1 \in F_1, \Psi(f_1) = f_0 \Leftrightarrow f_0\ (p) = \upsilon\ (\{\lambda_1(x) \mid x \in N(p)\}), \forall\ p \in R.$

3.2.1.3. Zonal operations

This is a special class of neighborhood operators, where one geo-field (usually a thematic map) is used as a spatial restriction on the operators on another geo-field (usually a DTM).

Definition 3.2.4 The zonal operation Z on a numerical geo-field f_1, defined by $\lambda_1: R \rightarrow V_1$, (where V_1 is the set of reals), and a thematic geo-field f_2, defined by $\lambda_2: R \rightarrow V_2$, (where V_{21} is a discrete set $\{\ v_1, \ldots v_n\}$), and a local function υ is such that:

$Z(f_1) = f_{new} \mid \lambda_{new}\ (p) = \upsilon\ (\lambda_1\ (x), x \in L(p))$ and the zonal region $L(p)$ satisfies

$\forall\ p \in R, \exists\ L(p) \subset R \wedge p \in L(p)$, such that $f_2(x) = v_1 \mid \forall\ x \in L(p)$.

3.2.2. Geo-Objects Algebra

3.2.2.1. Spatial Relationships

In the proposed model, geo-objects are represented as 2D geometries (points, lines and regions). As the operations of the Geo-objects algebra may involve spatial restrictions, the model defines spatial relationships divided in following categories:

- topological relationships, such as "inside" and "adjacent to", which are invariant to rotation, translation, and scaling transformations.

- directional relationships, such as “above” and “beside”.
- metrical relationships, derived from the distance operations.

In the proposed model, only topological and metrical relationships on R2 are considered, based on the following definitions:

- An area A is a 2D set of points of dimension 2, whose interior A^o is connected (with no holes) and which has a connected frontier δA.
- A line L is a set of connected points of dimension 1, whose frontier δL is the first and the last point or an empty set in the case of a circular line (an “island”), and its interior L^o is the set of the other points.
- A point P is a set of dimension 0, whose interior L^o is the point itself and whose frontier δP is empty.

To analyze the topological relationships on R2, this model considers the dimension of the intersection between the two sets, with a minimal set of five relationships ($touch$, in, $cross$, $overlap$ and $disjoint$) which are applicable to all cases. The formal definitions of these relationships are given below.
The $\boldsymbol{touch}$ relationship is applicable to area-area, line-area, line-line, point-area and point-line situations. A set of points S_1 *touches* another set S_2 when they have points in common, but their interiors do not:

$$S_1\ touch\ S_2 \Leftrightarrow\ (S_1\ \cap\ S_2\ \neq\ \emptyset)\ \wedge (S_1^o \cap\ S_2^o\ =\ \emptyset)$$

The $\boldsymbol{in}$ relationship is applicable to area-area, line-area, point-area and point-line situations. A set of points is *in* another when their intersection is the first set:
$S_1\ in\ S_2 \Leftrightarrow\ S_1\ \cap\ S_2\ =\ S1.$
The $\boldsymbol{cross}$ relationship is applicable in the case of line-line and line-area situations.
A line L *crosses* an area A when their interiors meet and the intersection of the two sets is not the line itself; two lines *cross* when their interiors have a nonempty intersection and this intersections is a set of points of dimension 0:
$L\ cross\ A\ \Leftrightarrow\ (L^o \cap\ A^o\ \neq\ \emptyset)\ \wedge\ ((L\ \cap\ A)\ \neq\ L).$
$L_1\ crossL_2 \Leftrightarrow\ (L_1^o \cap L_1^o\ \neq\ \emptyset)\ \wedge\ (dim\ (L_1 \cap L_2\ )\ =\ 0.$
The $\boldsymbol{overlap}$ relationship is applicable to area-area, line-line and point-point situations.
Two point sets S_1 and S_2 *overlap* when their intersection is different from them, but forms a set of points of the same dimension:
$S_1\ overlap\ S_2 \Leftrightarrow\ (S_1 \cap\ S_2 \neq\ S_1\ ) \wedge (S_1 \cap\ S_2 \neq\ S_2\ ) \wedge (dim\ (S_1^o \cap S_2^o)\ =\ dim(S_1^o)).$

3.2.2.2. Spatial operations

In order to define the spatial operations over geo-objects, it is necessary to establish the notion of a ***computable spatial predicate***.
Definition 3.2.5 Let R be a geographical region, and GO a set of geo-objects which have representations in R, defined by an object map $om\ = [R, GO, geo]$. A ***computable spatial predicate*** **x** is a spatial restriction, defined by a topological relationship (*inside, touch*, *cross, overlap* and *disjoint*) or a metrical relationship, which can be computed over the representations $geo(go_i)$ of the geo-objects $go_i\ \in\ GO$.

3.2.2.3. Spatial selection

Definition 3.2.6 Let R be a geographical region, GO a set of geo-objects and mo an object-map $mo = [R, GO, geo_1]$ which contains the spatial location of the geo-objects $go\ \in\ GO$ in R.
The *spatial selection* operation $\varphi :\ GO \rightarrow GO$, given a spatial predicate **x** which relates the geo-objects $go\ \in\ GO$ to a geo-object go^* which is represented in mo by a mapping $geo_2(go^*)$:
$\varphi_\xi(GO)\ =\ \{\ go\ \in\ GO\ |\ \xi(geo(go))\ \}.$
The output of such operation is a subset of the original set, composed of all geo-objects that satisfy the geometrical predicate, as in the following example:
· “select all counties of Srbija which are adjacent to the Severna Bačka municipalities (which contains the city of Subotica)”.

3.2.2.4. Spatial Join

Definition 3.2.7 Let R be a geographical region, GO_1 and GO_2 two sets of geo-objects and mo_1 and mo_2 object-maps $mo_i = [R, GO_i, geo_i]$ which contain, respectively, the spatial location of the geo-objects $go_1 \in GO_1$ and $go_2 \in GO_2$ in R. Let **x** be a spatial predicate computable for every pair of geographical locations $((geo_1(go_1), geo_2(go_2))$. The spatial join operation $\theta: GO_1 \times GO_2 \rightarrow GO_1 \times GO_2$ is such that:

$$\theta_\xi\ (GO_1, GO_2) = \{\ (go_1,\ go_2) \in (GO1, GO2)\ |\ \xi\ ((geo_1(go_1), geo_2(go_2)\ )\ \}.$$

The spatial join is an operation where a comparison between two sets of geo-objects GO_1 and GO_2 takes place, based on a spatial predicate which is computed over the representation of these sets. The name "spatial join" is employed by analogy to the join operation in relational algebra. The result of the spatial join operation is a set of object-pairs, which satisfy the spatial restriction. Examples are:

- "Find all villages located closer than 50 km to the main roads in Vojvodina".
- "Find all cities in the Province of Vojvodina which are located closer than 10 km from a water reservoir."

In the first example, the answer is a set of pairs of geo-objects (village, road) and in the second a set of pairs (cities, water reservoir).

3.2.3. Transformations between Geo-Fields and Geo-objects

Another set of operations for geographical data concerns the transformations that generate geo-fields from sets of geo-objects (and vice-versa). These transformation operations are of special importance, as they are the link between the two general classes of geographical data.

3.2.3.1. Generation of Geo-Objects from Geo-Fields

As an example of one important instance of such operations, we shall present spatial interpolation.

Definition 3.2.8 Let R be a geographical region, $V_1, V_2, .. V_n$ sets which define possible ranges for geo-fields, and F_i $(i = 1, .., n)$ be the class of all geo-fields which have R as a location and V_i as its range. Let GO be a set of geo-objects and mo be an object-map $mo = [R, GO, geo]$ which assigns geographical locations in R to the geo-objects in GO.

The spatial interpolation operation $\otimes: F_1 \times \ldots \times F_n \rightarrow GO$ is such that:

$$\forall f_1 \in F_1, \ldots, f_n \in F_n,$$

$\otimes(f_1, f_2, \ldots, f_n) = GO \Leftrightarrow \forall go \in GO, go = [\ v_1, \ldots, v_m, a_{m+1}, .., a_n, geo(go)]$, and
$geo\ (go) = \{\ p \in R\ | f_1(p) = v_1 \wedge \ldots \wedge f_n(p) = v_n\ \}$.

This definition corresponds to the generation of an object map from the spatial intersection of a set of geo-fields. This situation occurs, for example, in zoning applications, when an overlay of thematic maps is performed to obtain homogeneous zones. When a cadastral map is created from an overlay of geo-fields, each resulting geo-object inherits all descriptive attributes from the original geo-fields.

One example of interpolation is: Determine the homogeneous regions of Vojvodina as the intersection of the vegetation, geomorphology, and soils maps.

3.2.3.2. Generation of Geo-Fields from Geo-Objects

These operations take as input a set of geo-objects GO, represented in the geo-objects map mo and generate as output a field f_1, defined on a map M by a mapping $\lambda: M \rightarrow V$. We shall consider two operations, that of *distance maps* (buffer zones) and that of *attribute reclassification*.

Buffer zones operation

Definition 3.2.9. Let R be a geographical region, F a set of geo-fields defined over R whose range is $+$. Let GO be a set of geoobjects, and mo an object-map $mo = [R, GO, geo]$, which assigns geographical locations in R to the geo-objects in GO.

The buffer zones operation $\Delta: GO \rightarrow G$ induced by mo is such that, given a distance metric $dist$ computable in mo and an object $go \in GO$:
$\Delta_{mo}(go) = f \Leftrightarrow \forall p \in R, f(p) = dist(p, geo(go))$.

Attribute reclassification operation

Definition 3.2.10 Let R be a geographical region, GO be a set of geo-objects whose descriptive attributes are contained in $D(A_1) \times \ldots \times D(A_n)$, and mo an object-map $mo = [R, GO, geo]$, which assigns geographical locations in R to the geo-objects in GO.
Let F a set of geo-fields defined over R whose range is $D(A_i)$, where A_i is the i −th descriptive attribute of GO.
The attribute reclassification operation $\Omega: GO \rightarrow F$ induced by mo is such that:
$\Omega_{mo}(GO) = f_0 \Leftrightarrow (\forall\, go \in GO,\ f_0\,(geo(go)) = go.A_i)$.

4. Class model of the complex fuzzy spatial object

Atlas Platform class model of the complex spatial object relies upon two basic models:

- Linear fuzzy space model proposed in [1, 3],
- Formal object model of the geographic data proposed in [2].

The fundamental concept of the complex spatial object in our model is the *composite object*.

4.1. Composite object

An object *o* can be constructed out of other objects $o_1, ...o_n$, in which case *o* is called *composite* and $o_1, ...o_n$ are called the *components* of *o*. If an object is not composite, then it is called *simple*.
In our model, *classes* can be structured into hierarchies; the ancestors of a class C in the hierarchy are called the *superclasses* of C.
The real world is modelled as a collection of object-oriented classes, classified as *conventional classes* and *spatial classes* (i.e., *geo-classes*). The spatial classes model spatial fields and objects, while the conventional classes correspond to classes whose instances are non-spatial objects.
The same entity might be modelled as a part of spatial class or as part of a standard class, depending on the situation.
Each spatial class of objects has both locational and conventional attributes.
The *conventional* attributes are assumed to be derived from a universe ***U*** of *descriptive* attributes $\{A_1, \ldots, A_n\}$, defined on domains $D(A_1), \ldots, D(A_n)$.

4.1.1. Basic model hierarchy

The basic model hierarchy recognizes following classes: Fuzzy Spatial Region, Fuzzy Spatial Field, Fuzzy Spatial Object, and Fuzzy Spatial Object Map.

Fuzzy Spatial Region

Definition 4.1. Fuzzy polygon in $\mathbb{R}^2$ linear fuzzy space is called a **fuzzy spatial region**.

Fuzzy Spatial Field

A fuzzy spatial field represents a continuous spatial variable over some fuzzy region of the space.
Definition 4.2. Let R be a fuzzy spatial region. A **fuzzy spatial field** f is an object $[a_1, \ldots a_n, \ \lambda\ ]$, where $a_i \in D(Ai)$ and $\lambda: \ R \ \rightarrow \ V$ defines a fuzzy mapping between fuzzy points in R and fuzzy values on a domain V.
The fuzzy spatial fields can also be specialized. Depending on the range of the variable, the following subclasses of GEO-FIELD are defined analogously to the original object model of geographic data:

- NUMERICAL - an instance of this class defines a mapping λ: $R \ \rightarrow \ V$ such that V is the set of fuzzy real numbers.
- LINGUISTIC – an instance of this class defines a mapping λ: $R \ \rightarrow \ V$ such that V is the set of linguistic variables defined by corresponding fuzzy sets.

Fuzzy Spatial Object

Fuzzy spatial objects represent individualizable entities of some spatially determined domain. They are phenomena that may have one or more *graphical representations,* which correspond to the (in the case of environmental science most often geo-referenced) set of coordinates that describe the object's location.
Definition 4.3. Given a set of fuzzy regions $R_1, \ldots R_n$, a **fuzzy spatial object** go is an object $[a_1, \ldots a_n, geo_1, \ldots, geo_n]$, composed by the values $a_i \in D(Ai)$ and by a set of spatial locations geo_i (where $geo_i \ \subseteq \ R_i$).

Fuzzy Object Maps

Definition 4.4. Let R be a fuzzy region. A **fuzzy object map** mo is an object $[R, GO, geo]$ such that GO is a set of fuzzy spatial objects and geo is a mapping $GO \to R$, which assigns, for each fuzzy spatial object $go \in GO$, a fuzzy location $geo(go)$ in R.

4.1.2. Operations on fuzzy spatial data

In our model we define two main types of spatial algebras, defining operations on spatial data:

- Fuzzy Fields algebra: operations aimed at manipulation of fuzzy fields.
- Fuzzy Spatial Objects algebra: operations aimed at querying spatial objects in terms of descriptive and spatial properties.

4.1.2.1. Fuzzy Fields algebra

There are two basic types of fuzzy field operators in our model: **point** and **neighbourhood**.

Point operators

Based on input fuzzy spatial fields, this operator produces a new fuzzy spatial field with value in each point p depending only on the values in p in the input spatial fields. Point operations are specified as a mapping between the ranges of the input and output fields. Formal definition follows.

Definition 4.5. Let R be a fuzzy spatial region, $V_1, V_2, .. V_{n+1}$ be the sets which define possible ranges for fuzzy spatial fields, and F_i $(i = 1, .., n + 1)$ be the class of all fuzzy spatial fields which have R as a location and V_i as its range. The **point operation** Π: $F_1 \times F_2 \rightarrow F_{n+1}$ induces a function p such that, for every fuzzy spatial field $f_i \in F_i$ $(i = 1, .., n)$:

$$f_{n+1}(p) = p\,(f_1(p), \dots, f_n(p)), \forall p \in R$$

where the spatial values of the output fuzzy spatial field $f_{n+1} \in F_{n+1}$ are defined by the mapping $\lambda_{n+1}: M \to V_{n+1}$.

Neighbourhood operators

In this class of operators, the output fuzzy field is computed based on the values of a continuously-varying surface in the neighbourhood of each location of the input fuzzy field. What follows is the formal definition of the neighbourhood operations on fuzzy spatial fields preceded by the definition of the concept neighbourhood in fuzzy spatial region.

Definition 4.6. Given a fuzzy region R, a set of $P \subseteq R$ is said to be *connected* iff, for any two fuzzy points $p_1, p_2 \in P$ there is a fuzzy line connecting these two points which is fuzzy-entirely contained in R. A ***neighbourhood*** in R is a mapping $N: R \to 2^R$, such that $\forall p \in R,\ p \in N(p)$ and $N(p)$ are fuzzy connected.

Definition 4.7. Let R be a fuzzy region and F_0 and F_1 the sets of fuzzy spatial fields which are defined over R and whose range is $V_i, i = 0, 1$. Let $N: R \to 2^R$ and $\upsilon: 2^{V_1} \to V_0$. The neighbourhood operation $\Psi: F_1 \to F_0$ *induced* by υ is such that:

$$\forall f_1 \in F_1, \Psi(f_1) = f_0 \Leftrightarrow f_0(p) = \upsilon(\{\lambda_1(x) \mid x \in N(p)\}), \forall p \in R.$$

4.1.2.2. Fuzzy spatial objects algebra

Spatial Relationships

In the proposed model, fuzzy spatial objects are represented as 2D fuzzy geometries (points, lines and regions). As the operations of the fuzzy spatial objects algebra may involve spatial restrictions, the model defines spatial relationships divided in following categories:

- Fuzzy topological relationships, such as “fuzzy inside” and “fuzzy adjacent to”, which are invariant to transformations fuzzy rotation, fuzzy translation, and fuzzy scaling transformations.
- Fuzzy directional relationships, such as “fuzzy above”, “fuzzy bellow”, “fuzzy left”, “fuzzy right”, and “fuzzy beside”.
- Fuzzy metrical relationships, derived from the fuzzy distance operations.

In our model, topological and metrical relationships on R^2 are considered, based on the following definitions:

- A fuzzy area A is one of the following: a fuzzy point, a fuzzy line, a fuzzy circle or a fuzzy polygon, in linear fuzzy space.
- A fuzzy line L is a fuzzy line in linear fuzzy space.
- A fuzzy point P is a fuzzy point in linear fuzzy space.

So far, we propose a minimal set of two relationships ($\boldsymbol{in}$ and $\boldsymbol{cross}$) which are applicable to all cases. The formal definitions of these relationships are given below.
The $\boldsymbol{in}$ **relationship** is applicable to cases point-point, area-area, line-area, point-area and point-line situations. A fuzzy spatial object S_1 is $\boldsymbol{in}$ a fuzzy spatial object S_2 when following holds:

- **point-point situation**: Objects S_1 and S_2 satisfy spatial relation $\boldsymbol{coincide}$ in linear fuzzy space.
- **point-line situation**: Objects S_1 and S_2 satisfy spatial relation $\boldsymbol{contain}$ in linear fuzzy space.
- **point-area situation**: Fuzzy point S_1 is in fuzzy polygon S_2 if the core of the point S_1 is inside the fuzzy polygon S_2 including the support of the fuzzy polygon S_2.
- **line-area situation**: Fuzzy line S_1 is in fuzzy polygon S_2 if the start and end points of the line S_1 are completely inside the fuzzy polygon S_2 including the support of the S_2.
- **area-area situation**: Fuzzy polygon S_1 is in fuzzy polygon S_2 if the support of the fuzzy polygon S_1 is completely inside the fuzzy polygon S_2 including the support of the S_2 .

The $\boldsymbol{cross}$ relationship is applicable in the case of line-line and line-area situations.

- Line-line situation: Line S_1 crosses a line S_2 if
  - Start and end points of the lines S_1 and S_2 do not satisfy relation $\boldsymbol{colinear}$ in linear fuzzy space.
  - There is at least one point with a core in the fuzzy set S_1 and in the fuzzy set S_2.

Spatial operations

To define the spatial operations over fuzzy spatial objects, we first define the notion of a ***computable fuzzy spatial predicate***.
Definition 4.8. Let R be a fuzzy region, and GO a set of fuzzy spatial objects which have representations in R, defined by an object map $om = [R, GO, geo]$. A ***computable fuzzy spatial predicate*** **x** is a spatial restriction, defined by a topological relationship (*fuzzy inside*, *fuzzy cross*) or a metrical relationship as defined in fuzzy linear space, which can be computed over the representations $geo(go_i)$ of the fuzzy spatial objects $go_i \in GO$.
Next, we define two fuzzy spatial operations: *fuzzy spatial selection* and *fuzzy spatial join*.

Fuzzy spatial selection

Definition 4.9. Let R be a fuzzy region, GO a set of fuzzy spatial objects and mo an fuzzy spatial object map $mo = [R, GO, geo_1]$ which contains the fuzzy spatial location of the spatial objects $go \in GO$ in R.
The *fuzzy spatial selection* operation $\varphi : GO \rightarrow GO$, given a spatial predicate **x** which relates the fuzzy spatial objects $go \in GO$ to a fuzzy spatial object go^* which is represented in mo by a mapping $geo_2(go^*)$ is:
$\varphi_\xi(GO) = \{ go \in GO \mid \xi(geo(go)) \}$.
The output of such operation is a fuzzy subset of the original fuzzy set, composed of all fuzzy spatial objects that satisfy the geometrical predicate, as in the following example:
· “select all counties of Srbija which are to some extent adjacent to the Severna Bačka municipalities (which contains to some extent the city of Subotica)”.

Fuzzy spatial join

Definition 4.10. Let R be a fuzzy region, let GO_1 and GO_2 be two sets of fuzzy spatial objects, and let mo_1 and mo_2 be fuzzy spatial object maps $mo_i = [R, GO_i, geo_i]$ which contain, respectively, the fuzzy spatial location of the spatial fuzzy objects $go_1 \in GO_1$ and $go_2 \in GO_2$ in R. Let **x** be a fuzzy spatial predicate computable for every pair of spatial locations $((geo_1(go_1), geo_2(go_2))$. The spatial join operation θ: $GO_1 \times GO_2 \rightarrow GO_1 \times GO_2$ is such that:

$$\theta_\xi\ (GO_1, GO_2) = \{ (go_1,\ go_2) \in (GO1, GO2) \mid \xi\ ((geo_1(go_1), geo_2(go_2)\ )\ \}.$$

The spatial join is an operation where two sets of fuzzy spatial objects GO_1 and GO_2 are compared, based on a fuzzy spatial predicate computed over the representation of these sets. The result is a set of object pairs, which satisfy the spatial restriction to some extent while, at the same time, objects individually satisfy spatial location restriction to some extent. Examples are:

- “Find all air quality measurement facilities located **very near** to the **main roads** in Vojvodina”.

In general case, the answer to this query is a set of pairs of spatial objects (facilities, roads) where *facilities* is an object that is air quality measurement facilities to some extent, located in Vojvodina to some extent, roads is an object that is main road to some extent also located in Vojvodina to some extent and, finally, those two objects are very near to each other to some extent.

5. Concluding remarks

This report proposes a conceptual model of the complex fuzzy spatial objects within the ATLAS platform. The proposed model provides for representing composite, highly complex spatial objects described with spatial and non-spatial properties that include uncertainty, imprecision, and vagueness. The proposed model relies on object paradigm, more precisely class pattern.

The indisputable advantage of the proposed model is its ability to incorporate uncertainty and imprecision when describing spatial and non-spatial properties of the system under modelling.

From the other hand, reliance on the class pattern could be considered a notable disadvantage that introduces serious restrictions on composite structure. Therefore, we are planning further research that will replace class-based inheritance with a prototype-based one.

6. References

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