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ATLAS

Artificial Intelligence Theoretical Foundations for Advanced Spatio-Temporal Modelling of Data and Processes

Task 3: Computational Intelligence base: Theoretical foundations

Report for the deliverable No. 3.1

Mathematical models of fuzzy spatial primitives

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Report prepared by (alphabetic order):
Đorđe Obradović (US), Zora Konjović (US), Endre Pap (US)

Report reviewed internally by: Andreja Stojić (IPB)

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Summary

This report presents the mathematical models of basic planar imprecise geometric objects (fuzzy line, fuzzy triangle and fuzzy circle), as well as the basic measurement functions (distance between fuzzy point and fuzzy line, fuzzy point and fuzzy triangle, two fuzzy lines and two fuzzy triangles), spatial operations (linear combination of two fuzzy points), and main spatial relations (coincidence, between and collinear). These models are the theoretical foundation for spatial modelling in ATLAS platform.

Keywords

Fuzzy point, fuzzy line, fuzzy triangle, fuzzy collinear, linear fuzzy space

1. Introduction

Extraction of basic geometric features, such as lines, from digital raster image is one of fundamental processes in image analysis. Common problems in image analysis arise from the fact that a discrete space (digital raster image) is used for real-world's elements representation, while spatial relations apply rules of continual space. For example, line is represented as a set of discrete points that usually do not have to be collinear, which contradicts the line definition. Real-world objects are mapped to the digital raster image through a variety of sensors, making the image only an approximation to the real-world object. Due to imperfections in either the image data or the edge detector, there may be missing points or pixels on lines as well as spatial deviations between ideal line and the set of imprecise points obtained from the edge detector. The overall effect is an image that has some distortion in its geometry.

The research topic which is relevant for this paper is modeling of basic planar imprecise geometry objects and their relations, as well as their application to spatial data management systems. In the sequel, the results belonging to this topic will be briefly presented.

An overview of the papers dealing with imprecise point objects modeling is given in [1] and [2].

This report consists of six sections. Following this introduction, several definitions and preliminaries related to imprecise point object model are presented in Section II. Mathematical models of the basic planar imprecise geometrical objects are set out in Section III. Section IV contains basic spatial measurement functions and section V contains basic spatial relations. Section VI contains concluding remarks.

2. Preliminaries

Definition 2.1 *Fuzzy point* $P \in \mathbb{R}^2$, denoted by $\tilde{P}$ is defined by its membership function $\mu_{\tilde{P}} \in \mathcal{F}^2$**,** where the set $\mathcal{F}^2$ contains all membership functions $u: \mathbb{R}^2 \to [0,1]$ satisfying following conditions:

i) $(\forall u \in \mathcal{F}^2)(\exists_1 P \in \mathbb{R}^2)\, u(P) = 1$,
ii) $(\forall X_1, X_2 \in \mathbb{R}^2)(\lambda \in [0,1])\, u(\lambda X_1 + (1-\lambda)X_2) \geq \min\big(u(X_1), u(X_2)\big)$,
iii) function u is upper semi continuous,
iv) $[u]^\alpha = \{X | X \in \mathbb{R}^2, u(X) \geq \alpha\}$ α-cut of function u is convex.

The point from $\mathbb{R}^2$, with membership function $\mu_{\tilde{P}}(P) = 1$, will be denoted by P (P is the core of the fuzzy point $\tilde{P}$), and the membership function of the point $\tilde{P}$ will be denoted by $\mu_{\tilde{P}}$. By $[P]^\alpha$ we denote the α-cut of the fuzzy point (this is a set from $\mathbb{R}^2$).

Definition 2.2 $\mathbb{R}^2$ *Linear fuzzy space* is the set $\mathcal{H}^2 \subset \mathcal{F}^2$ of all functions which, in addition to the properties given in Definition 2.1, are:

i) Symmetric against the core $S \in \mathbb{R}^2$
$(\mu(S) = 1)$,
$\mu(V) = \mu(M) \wedge \mu(M) \neq 0 \Rightarrow d(S,V) = d(S,M)$,
where $d(S,M)$ is the distance in $\mathbb{R}^2$.

ii) Inverse-linear decreasing w.r.t. points' distance from the core according to:
If $r \neq 0$

$$\mu_{\tilde{S}}(V) = \max\left(0, 1 - \frac{d(S,V)}{|r_S|}\right),$$

if $r = 0$

$$\mu_{\tilde{S}}(V) = \begin{cases} 1 & if \quad S = V \\ 0 & if \quad S \neq V\ , \end{cases}$$

where $d(S,V)$ is the distance between the point V and the core S ($V, S \in R^n$) and $r \in \mathbb{R}$ is constant.

Elements of that space are represented as ordered pairs $\tilde{S} = (S, r_S)$ where $S \in \mathbb{R}^2$ is the core of $\tilde{S}$, and $r_S \in \mathbb{R}$ is the distance from the core for which the function value becomes 0; in the sequel parameter r_S will be denoted as *fuzzy support radius*.

Definition 2.3 Let the linear fuzzy space be defined on . Fuzzy relations $\leq^{RF}$ and $\leq^{LF}$ for the set are defined by membership functions

$$\mu(\tilde{A} \leq^{RF} \tilde{B}) = \begin{cases} 0 & if \quad A > B, \\ \frac{B-A}{r_A - r_B} & if \quad A \leq B \wedge A + r_A > B + r_B \\ 1 & if \quad A \leq B \wedge A + r_A \leq B + r_B\ , \end{cases}$$

$$\mu(\tilde{A} \leq^{LF} \tilde{B}) = \begin{cases} 0 & if \quad A > B \\ \dfrac{B-A}{r_B - r_A} & if \quad A \leq B \wedge\ A - r_A > B - r_B \\ 1 & if \quad A \leq B \wedge A - r_A \leq B - r_B, \end{cases}$$

respectively, where $\tilde{A} = (A, r_A)$ and $\tilde{B} = (B, r_B)$ are points from , A is the core of $\tilde{A}$ and r_A is a parameter determining the membership function of point $\tilde{A}$.

3. Basic fuzzy plane geometry objects in R^2 linear fuzzy space

In this section we present theoretical models of basic operations over linear fuzzy space $\mathcal{H}^2$ defined on $\mathbb{R}^2$, as well as their properties which will be used in definitions of basic fuzzy plane geometry objects.

Definition 3.1 Let $\tilde{A}, \tilde{B} \in \mathcal{H}^2$. An operator $+ : \mathcal{H}^2 \times \mathcal{H}^2 \rightarrow \mathcal{H}^2$ is called *fuzzy points addition* given by

$$\tilde{A} + \tilde{B} = (A + B, r_A + r_B),$$

where $A + B$ is a vector addition, and $r_A + r_B$ is a scalar addition.

Definition 3.2 Let $\mathcal{H}^2$ be a linear fuzzy space. Then a function $f : \mathcal{H}^2 \times \mathcal{H}^2 \times [0,1] \rightarrow \mathcal{H}^2$ is called *linear combination* of the fuzzy points $\tilde{A}, \tilde{B} \in \mathcal{H}^2$ given by

$$f(\tilde{A}, \tilde{B}, u) = \tilde{A} + u \cdot (\tilde{B} - \tilde{A}),$$

where $u \in [0,1]$ and operator is a scalar multiplication of fuzzy point.

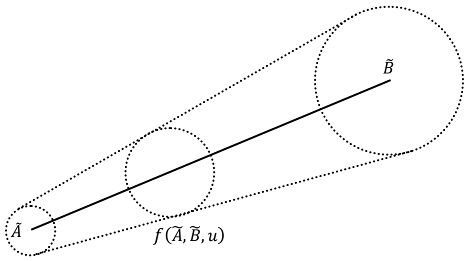

Figure 3.1 Geometric illustration of linear combination of the fuzzy points $\tilde{A},\tilde{B} \in \mathcal{H}^2$

Figure 3.1 shows alpha-cuts of the fuzzy points $\tilde{A},\tilde{B} \in \mathcal{H}^2$ and alpha-cut of the fuzzy point $f(\tilde{A},\tilde{B},u)$. Cores of the all three fuzzy points are collinear. All straight lines that connects points $X \in [\tilde{A}]^\alpha$ and $Y \in [\tilde{B}]^\alpha$ always contains at least one point from $[f(\tilde{A},\tilde{B},u)]^\alpha$.

Remark. The linear combination of the two fuzzy points could be also expressed as

$$f(\tilde{A},\tilde{B},u) = (1-u)\tilde{A} + u \cdot \tilde{B}.$$

Definition 3.3 Let $\tilde{A},\tilde{B} \in \mathcal{H}^2$ and $\tilde{A} \neq \tilde{B}$. Then a point $T_{AB} \in \mathbb{R}^2$ is called *internal homothetic center* if the following holds

$$T_{AB} = A + \frac{a_r}{a_r+b_r}(B-A),$$

where $\tilde{A} = (A, a_r)$ and $\tilde{B} = (B, b_r)$.

In the previous definitions we introduced the basic element of a linear fuzzy space and provide an overview of its basic features. Fuzzy points are used to describe the position of a real object when there is some uncertainty to the measured position. Most often this uncertainty in practical applications is ignored. There are applications in which real objects are not only represented by the position but the entire series of uniformly spaced points. These points can be distributed along a curve that has a beginning and an end. Curve that connects two points is called a line or path.
If the points that represent the path are imprecise, then the whole line should be described in way similar to imprecise point's description. In this section we will present mathematical model for such fuzzy line.

Definition 3.4 Let $\mathcal{H}^2$ be a linear fuzzy space and function f is a linear combination of the fuzzy points $\tilde{A}$ and $\tilde{B}$. Then a fuzzy set $\widetilde{AB}$ is called *fuzzy line* if following holds

$$\widetilde{AB} = \bigcup_{u\in[0,1]} f(\tilde{A},\tilde{B},u).$$

Theorem 3.1 Let $\mathcal{H}^2$ be linear fuzzy space, fuzzy line $\widetilde{AB}$ defined by fuzzy points $\tilde{A}$ and $\tilde{B} \in \mathcal{H}^2$. Then following holds

$$\widetilde{AB} = \widetilde{BA}.$$

Definition 3.5 Let $\widetilde{AB}$ be *fuzzy line* defined on linear fuzzy space $\mathcal{H}^2$ and $X \in \mathbb{R}^2$. Then a fuzzy point $\widetilde{X'} \subset \widetilde{AB}$ is called *fuzzy image of point X on fuzzy line* $\widetilde{AB}$, and a real number $u \in [0,1]$ is called *eigenvalue of the fuzzy image X* on fuzzy line $\widetilde{AB}$ if following hold

(i) $\widetilde{X'} = \tilde{A} + u(\tilde{B} - \tilde{A})$,
(ii) $d\left(X, [\widetilde{X'}]^1\right) = \min\{d(X,Y) | \forall Y \in [\widetilde{AB}]^1\}$,
(iii) $u = \min\left(1, max\left(0, \frac{(x_1-a_1)(b_1-a_1)+(x_2-a_2)(b_2-a_2)}{(b_1-a_1)^2+(b_2-a_2)^2}\right)\right)$,

where $X = (x_1,x_2)$, $\tilde{A} = ((a_1,a_2),a_r)$ i $\tilde{B} = ((b_1,b_2),b_r)$.

Remark. If the eigenvalue of the fuzzy image X is equal 0, then fuzzy image is the starting fuzzy point, if eigenvalue is equal 1 it is the final point, otherwise it is the inner point of a fuzzy line.

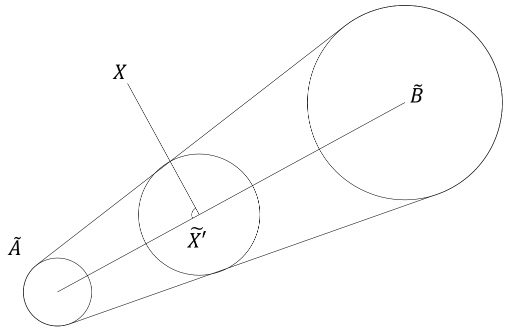

Figure 3.2 Geometric illustration of the fuzzy image of point X on fuzzy line $\widetilde{AB}$

Figure 3.2 shows alpha cuts of the fuzzy points $\tilde{A}, \tilde{B} \in \mathcal{H}^2$ and alpha-cut of the fuzzy image of point X on fuzzy line $\widetilde{AB}$ (fuzzy point $\widetilde{X}'$). The nearest point from point X to the straight line $\overline{AB} = [\widetilde{AB}]^0$ is core of its fuzzy image $\widetilde{X}'$.

Theorem 3.2 Let $\widetilde{AB} \in L^2$ be fuzzy line, $\widetilde{X}' \in \mathcal{H}^2$ fuzzy image of point $X \in \mathbb{R}^2$ on $\widetilde{AB}$ and $u \in [0,1]$ eigenvalue of the fuzzy image X on $\widetilde{AB}$. Then point X belongs to fuzzy set $\widetilde{AB}$ according to following

$$\mu_{\widetilde{AB}}(X) = \begin{cases} \mu_{\tilde{A}}(X) & if \quad u_m = 0 \\ \mu_{\tilde{X}_T}(X) & if\ 0 < u_m < 1 \\ \mu_{\tilde{B}}(X) & if \quad u_m = 1 \end{cases},$$

where fuzzy point $\tilde{X}_T = \tilde{A} + u_m(\tilde{B} - \tilde{A})$ and $u_m = u + \frac{(b_r - a_r)}{x_r'} \frac{d(X,X')^2}{d(A,B)^2}$.

Definition 3.6 Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{H}^2$ be fuzzy points with noncollinear cores ($\tilde{A} \neq \tilde{B} \neq \tilde{C}$) and function f is a linear combination of two fuzzy points. Then the fuzzy set $\widetilde{ABC}$ is called *fuzzy triangle*, if the following holds

$$\widetilde{ABC} = \bigcup_{u=0}^{1} f(\tilde{A}, \bigcup_{v=0}^{1} f(\tilde{B}, \tilde{C}, v), u)$$

It's membership function is denoted by $\mu_{\widetilde{ABC}}(X)$ and determined according to the following formula

$$\mu_{\widetilde{ABC}}(X) = \max_{u \in [0,1], v \in [0,1]} \{\mu_{\tilde{Y}}(X) | \tilde{Y} = f(\tilde{A}, f(\tilde{B}, \tilde{C}, v), u)\}.$$

α-cut of fuzzy triangle $\widetilde{ABC}$ is denoted by $[\widetilde{ABC}]^\alpha$.

Figure 3.3 shows geometric illustration of the fuzzy triangle membership function and corresponding $\alpha - cuts$.

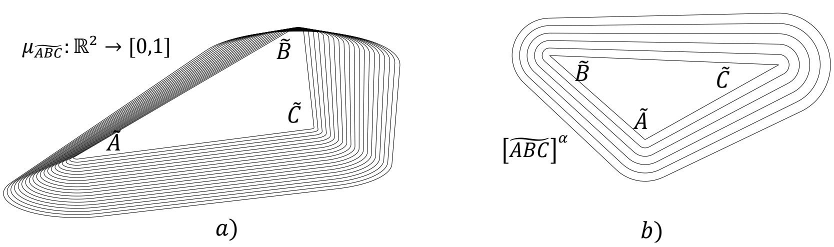

Figure 3.3: a) Fuzzy triangle membership function b) Fuzzy triangle $\alpha - cuts$

Definition 3.7 Let $\widetilde{ABC}$ be a fuzzy triangle defined on fuzzy linear space $\mathcal{H}^2$. Fuzzy point $\tilde{X} \subset \widetilde{ABC}$ is called *edge point of the fuzzy triangle* $\widetilde{ABC}$ if for all $\alpha \in [0,1]$ a point $Y \in [\tilde{X}]^\alpha$ exists such that all its neighborhoods contain at least one point from $[\widetilde{ABC}]^\alpha$ and at least one point outside of $[\widetilde{ABC}]^\alpha$.

Remark. α-cut of all edge points intersect α-cut of fuzzy triangle in at least one point.

Definition 3.8 Let $\widetilde{ABC}$ be a fuzzy triangle defined on fuzzy linear space $\mathcal{H}^2$. Fuzzy point $\tilde{X} \subset \widetilde{ABC}$ is called *inner point of fuzzy triangle* $\widetilde{ABC}$ if it is not an edge point.

Definition 3.9 Let $\widetilde{ABC}$ be a fuzzy triangle defined on fuzzy linear space $\mathcal{H}^2$. Union of all edge points of the fuzzy triangle $\widetilde{ABC}$ is called fuzzy edge of fuzzy triangle $\widetilde{ABC}$, denoted by $\partial\widetilde{ABC}$.

Theorem 3.3 Let $\widetilde{ABC}$ be a fuzzy triangle defined on $\mathcal{H}^2$. Then, for every fuzzy point $\tilde{X} \in \partial\widetilde{ABC}/\{\tilde{A},\tilde{B},\tilde{C}\}$ and $\alpha \in [0,1]$ the single point $T \in [\tilde{X}]^\alpha$ exists such that all its neighborhoods contain at least one point from $[\widetilde{ABC}]^\alpha$ and at least one point outside of $[\widetilde{ABC}]^\alpha$.

Theorem 3.4 Let $\widetilde{ABC}$ be a fuzzy triangle defined on linear fuzzy space $\mathcal{H}^2$. Then for all $X \in \mathbb{R}^2$ the following holds

$$\mu_{\widetilde{ABC}}(X) = \mu_{\widetilde{CAB}}(X) = \mu_{\widetilde{BCA}}(X).$$

Direct consequence of this proposition is that a fuzzy triangle can be represented by three fuzzy points, i.e., the set $\{\tilde{A},\tilde{B},\tilde{C}\}$.

Fuzzy circle is also one of the basic planar imprecise geometrical objects. Analogously to the definitions of fuzzy line and fuzzy triangle, which is an extension of a precise circle, we define a fuzzy circle as a union of fuzzy points. Thereby, we also take care that a newly defined geometrical object is appropriate for implementation in GIS applications.

Definition 3.10 Let be a fuzzy space defined on , fuzzy relation $\leq^{RF}$ be fuzzy ordering in linear fuzzy space , $C \in \mathbb{R}^2$ and $\tilde{R} \in \mathcal{H}$. Then the union of all fuzzy points $\tilde{A} \in \mathcal{H}^2$ such that

$$\mu\left(\tilde{d}\left(C,\tilde{A}\right) \leq^{RF} \tilde{R}\right) = 1,$$

is called *fuzzy circle* with center C and radius $\tilde{R}$.
Fuzzy circle is represented by the ordered pair $(C,\tilde{R})$.

Theorem 3.5 Let $\left(C,\tilde{R}\right)$ be a fuzzy circle defined on linear fuzzy space $\mathcal{H}^2$. Then the value of the fuzzy circle membership function in point $X \in R^2$ is determined according to the following formula

$$\mu_{(C,\tilde{R})}(X) = \max\left(0, min\left(1, 1 - \frac{d(X,C)-R}{r_r}\right)\right),$$

where $\tilde{R} = (R, r_r)$.

4. Spatial measurement in R^2 linear fuzzy space

Measurement of the space, especially a distance between plane geometry objects is defined as a generalization of the concept of physical distance. Distance function or metric is a function that behaves according to specific set of rules. In this section we present the basic distance functions between fuzzy plane geometry objects and their main properties according to the set of rules presented in papers [1], [20]

Definition 4.1 Let $\mathcal{H}^2$ be a linear fuzzy space, $\tilde{d}: \mathcal{H}^2 \times \mathcal{H}^2 \to \mathcal{H}^+$, $L, R: [0,1] \times [0,1] \to [0,1]$ be symmetric, associative and non-decreasing for both arguments, and $L(0,0) = 0$, $R(1,1) = 1$. The ordered quadruple $(\mathcal{H}^2, \tilde{d}, L, R)$ is called fuzzy metric space and the function $\tilde{d}$ is a *fuzzy metric*, if and only if the following conditions hold:

(i) $\tilde{d}(\tilde{X}, \tilde{Y}) = \tilde{0} \Leftrightarrow [\tilde{X}]^1 = [\tilde{Y}]^1$.

(ii) $\tilde{d}(\tilde{X}, \tilde{Y}) = \tilde{d}(\tilde{Y}, \tilde{X})$ for each $\tilde{X}, \tilde{Y} \in \mathcal{H}^2$.

(iii) $\forall \tilde{X}, \tilde{Y} \in \mathcal{H}^2$:

a. $\tilde{d}(\tilde{X}, \tilde{Y})(s+t) \geq L\big(d(x,z)(s), d(z,y)(t)\big)$ if
$s \leq \lambda_1(x,z) \wedge t \leq \lambda_1(z,y) \wedge s+t \leq \lambda_1(x,y)$

b. $\tilde{d}(\tilde{X}, \tilde{Y})(s+t) \leq R\big(d(x,z)(s), d(z,y)(t)\big)$ if
$s \geq \lambda_1(x,z) \wedge t \geq \lambda_1(z,y) \wedge s+t \geq \lambda_1(x,y)$,

where the α-cut of fuzzy number $\tilde{d}(x,y)$ is given by $\left[\tilde{d}(\tilde{X}, \tilde{Y})\right]^\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)]$ $(x, y \in \mathbb{R}^+, 0 < \alpha \leq 1)$. The fuzzy zero, $\tilde{0}$ is a *non-negative* fuzzy number with $[\tilde{0}]^1 = 0$.

Remark: Following distance functions are fuzzy metrics.

(i) $\tilde{d}(\tilde{X}, \tilde{Y}) =_{DF} (d(X,Y), (r_X + r_Y))$

(ii) $\tilde{d}(\tilde{X}, \tilde{Y}) =_{DF} (d(X,Y), max(r_X, r_Y))$

(iii) $\tilde{d}(\tilde{X}, \tilde{Y}) =_{DF} (d(X,Y), |r_X - r_Y|)$

Distance (iii) also satisfies set of rulles which define clasic metric.

In the following definitions we extend distance between fuzzy points to distance between different fuzzy plane geometric objects, such as distance between fuzzy point and fuzzy line, fuzzy point and fuzzy triangle and at last between two fuzzy triangles.

Definition 4.2 Let $\mathcal{H}^2$ be a linear fuzzy space, $\mathcal{L}^2$ set of all fuzzy linesdefined on $\mathcal{H}^2$, $\tilde{d}$ is fuzzy distance between fuzzy points, and μ_L is membership function of the fuzzy relation minimal (Definition 15. in the paper [1]). The function $dist: \mathcal{H}^2 \times \mathcal{L}^2 \to \mathcal{H}^+$ is called *distance between fuzzy point and fuzzy line* if the following holds:

$$dist(\tilde{T}, \widetilde{AB}) = \tilde{d}(\tilde{T}, \tilde{X})$$

where $\tilde{X} \in \widetilde{AB}$ such that $\mu_L\left(\tilde{d}(\tilde{T}, \tilde{X})\right) = hgt\big(\{\tilde{d}(\tilde{T}, \tilde{Y}) | \forall \tilde{Y} \in \widetilde{AB}\}\big)$.

Definition 4.3 Let $\mathcal{H}^2$ be linear fuzzy space, $\mathcal{T}^2$ be a set of all fuzzy triangles defined on $\mathcal{H}^2$, $\tilde{d}$ is fuzzy distance between fuzzy points and μ_L is membership function of the fuzzy relation minimal. The function $dist: \mathcal{H}^2 \times \mathcal{T}^2 \to \mathcal{H}^+$ is called distance between fuzzy point and fuzzy triangle if the following holds:

$$dist(\tilde{T}, \widetilde{ABC}) = \tilde{d}(\tilde{T}, \tilde{X})$$

where $\tilde{X} \in \widetilde{ABC}$ such that $\mu_L\left(\tilde{d}(\tilde{T}, \tilde{X})\right) = hgt\big(\{\tilde{d}(\tilde{T}, \tilde{Y}) | \forall \tilde{Y} \in \widetilde{ABC}\}\big)$

Definition 4.4 Let $\mathcal{H}^2$ be linear fuzzy space, $\mathcal{L}^2$ set of all fuzzy lines on $\mathcal{H}^2$ $\tilde{d}$ is fuzzy distance between fuzzy points and μ_L is membership function of the fuzzy relation minimal. Th function $dist: \mathcal{L}^2 \times \mathcal{L}^2 \to \mathcal{H}^+$ is called distance between two fuzzy lines if the following holds:

$$dist(\widetilde{AB}, \widetilde{CD}) = \tilde{d}(\tilde{X}, \tilde{Y})$$

where $\tilde{X} \in \widetilde{AB}$ and $\tilde{Y} \in \widetilde{CD}$ such that

$$\mu_L\left(\tilde{d}(\tilde{X}, \tilde{Y})\right) = hgt\big(\{\tilde{d}(\tilde{Q}, \tilde{W}) | \forall \tilde{Q} \in \widetilde{AB} \wedge \forall \tilde{W} \in \widetilde{CD}\}\big)$$

Definition 4.5 Let $\mathcal{H}^2$ be a linear fuzzy space, $\mathcal{L}^2$ be a set of all fuzzy lines on $\mathcal{H}^2$, $\mathcal{T}^2$ be a set of all fuzzy triangles, $\tilde{d}$ is fuzzy distance between fuzzy points and μ_L is membership function of the fuzzy relation minimal. The function $dist\colon \mathcal{L}^2 \times \mathcal{T}^2 \to \mathcal{H}^+$ is called distance between fuzzy line and fuzzy triangle if the following holds:

$$dist\left(\widetilde{AB}, \widetilde{CDE}\right) = \tilde{d}(\tilde{X}, \tilde{Y})$$

where $\tilde{X} \in \widetilde{AB}$ and $\tilde{Y} \in \widetilde{CDE}$ satisfies condition

$$\mu_L\left(\tilde{d}\left(\tilde{X}, \tilde{Y}\right)\right) = hgt\left(\left\{\tilde{d}\left(\tilde{Q}, \widetilde{W}\right) \middle| \forall \tilde{Q} \in \widetilde{AB} \wedge \forall \widetilde{W} \in \widetilde{CDE}\right\}\right)$$

Definition 4.6 Let $\mathcal{H}^2$ be linear fuzzy space, $\mathcal{T}^2$ be a set of all fuzzy triangles on $\mathcal{H}^2$, $\tilde{d}$ is fuzzy distance between fuzzy points and μ_L is membership function of the fuzzy relation minimal. The function $dist\colon \mathcal{T}^2 \times \mathcal{T}^2 \to \mathcal{H}^+$ is called distance between two fuzzy triangles if the following holds:

$$dist\left(\widetilde{ABC}, \widetilde{DEF}\right) = \tilde{d}(\tilde{X}, \tilde{Y})$$

where fuzzy points $\tilde{X} \in \widetilde{ABC}$ and $\tilde{Y} \in \widetilde{DEF}$ such that

$$\mu_L\left(\tilde{d}\left(\tilde{X}, \tilde{Y}\right)\right) = hgt\left(\left\{\tilde{d}\left(\tilde{Q}, \widetilde{W}\right) \middle| \forall \tilde{Q} \in \widetilde{ABC} \wedge \forall \widetilde{W} \in \widetilde{DEF}\right\}\right).$$

5. Spatial relations in $\boldsymbol{R^2}$ linear fuzzy space

Spatial relations (predicates) are functions that are used to establish mutual relations between the fuzzy geometric objects. The basic spatial relations are *coincide*, *between* and *collinear*. In this section we give their definitions and basic properties.

Fuzzy relation *coincidence* expresses the degree of truth that two fuzzy points are on the same place.

Definition 4.1 Let λ be the Lebesgue measure on the set $[0,1]$ and $\mathcal{H}^2$ is a linear fuzzy space. The fuzzy relation $coin\colon \mathcal{H}^2 \times \mathcal{H}^2 \to [0,1]$ is *fuzzy coincidence* represented by the following membership function

$$\mu_{coin}\left(\tilde{A}, \tilde{B}\right) = \lambda(\{\alpha \mid \left[\tilde{A}\right]^\alpha \cap [\tilde{B}]^\alpha \neq \emptyset\}).$$

Remark. Since the lowest α is always 0, then a membership function of the *fuzzy coincidence* is given by

$$\mu_{coin}\left(\tilde{A}, \tilde{B}\right) = max\{\alpha \mid \left[\tilde{A}\right]^\alpha \cap [\tilde{B}]^\alpha \neq \emptyset\}.$$

Proposition

"*Fuzzy point $\tilde{A}$ is coincident to fuzzy point $\tilde{B}$*"

is partially true with the truth degree $\mu_{coin}\left(\tilde{A}, \tilde{B}\right)$; in the Theorem 4.1 we present method for its calculation.

Theorem 4.1 Let the fuzzy relation *coin* be a fuzzy coincidence. Then the membership function of the fuzzy relation *fuzzy coincidence* is determined according to the following formula

$$\mu_{coin}\left(\tilde{A}, \tilde{B}\right) = \begin{cases} 0 & if \quad |a_r| + |b_r| = 0 \wedge d(A,B) \neq 0, \\ \max\left(0, 1 - \dfrac{d(A,B)}{|a_r| + |b_r|}\right) & if \quad |a_r| + |b_r| \neq 0, \\ 1 & if \quad |a_r| + |b_r| = 0 \wedge d(A,B) = 0\,. \end{cases}$$

Fuzzy relation *contains* or *between* is a measure that fuzzy point belongs to fuzzy line or fuzzy line contains fuzzy point.

Definition 4.2 Let λ be Lebesgue measure on the set $[0,1]$, $\mathcal{H}^2$ linear fuzzy space and $\mathcal{L}^2$ be set of all fuzzy lines defined on $\mathcal{H}^2$. Then fuzzy relation $contain: \mathcal{H}^2 \times \mathcal{L}^2 \to [0,1]$ is *fuzzy contain* represented by following membership function

$$\mu_{contain}(\tilde{A}, \widetilde{BC}) = \lambda(\{\alpha \mid [\tilde{A}]^\alpha \cap [\widetilde{BC}]^\alpha \neq \emptyset\}).$$

Remark. Its membership function could be also represented as

$$\mu_{contain}(\tilde{A}, \widetilde{BC}) = \lambda(\{\alpha | \exists u \in [0,1] \wedge \exists X \in [\tilde{A}]^\alpha \wedge \exists Y, Z \in [\widetilde{BC}]^\alpha \wedge \ X = Y + u(Z - Y)\}).$$

Proposition

"*Fuzzy line* $\widetilde{BC}$ *contain fuzzy point* $\tilde{A}$"

is partially true with the truth degree $\mu_{contain}(\tilde{A}, \widetilde{BC})$; in the Theorem 4.2 we present method for its efficient calculation.

Theorem 4.2 Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{H}^2$ be fuzzy points defined on $\mathcal{H}^2$ linear fuzzy space, $u \in [0,1]$ and $\widetilde{A'}$ be *fuzzy image of point A on fuzzy line* $\widetilde{BC}$. Points T_{AB} and T_{AC} are *internal homothetic center* fuzzy points for fuzzy points $\tilde{A}$ and $\tilde{B}$ and $\tilde{A}$ and $\tilde{C}$ respectively. Then the membership function of the fuzzy relation *fuzzy contain* is determined according to the following formula

$$\mu_{contain}(\tilde{A}, \widetilde{BC}) = \begin{cases} \mu_{coin}(\tilde{A}, \tilde{A}') \ ako\ je\ u \in \{0,1\} \\ \mu_{\tilde{A}}(A^*) ako\ je\ u \in (0,1) \end{cases},$$

where point A^* is a projection of core of $\tilde{A}$ on the line passing through the points T_{AB} and T_{AC}.

Collinearity is also one of the fundamental relations between three points in plane geometry. In the following definition we will present our definition of *fuzzy collinearity* in fuzzy linear space, as well as the method for its practical computation.

Definition 4.3 Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{H}^2$ be a fuzzy points defined on $\mathcal{H}^2$ linear fuzzy space and λ be Lebesgue measure on the set $[0,1]$. The fuzzy relation $coli: \mathcal{H}^2 \times \mathcal{H}^2 \times \mathcal{H}^2 \to [0,1]$ is *fuzzy collinearity* between three fuzzy points and it is represented by following membership function

$$\mu_{coli}(\tilde{A}, \tilde{B}, \tilde{C}) = \lambda\{\alpha | \exists u \in R \wedge \exists X \in [\tilde{A}]^\alpha \wedge \exists Y \in [\tilde{B}]^\alpha \wedge \ \exists Z \in [\tilde{C}]^\alpha \wedge \ \ A = B + u(C - B)\}\ .$$

Proposition

"Fuzzy points $\tilde{A}, \tilde{B}$ *and* $\tilde{C}$ *are collinear"*

is partially true with the truth degree $\mu_{coli}(\tilde{A}, \tilde{B}, \tilde{C})$; in the Theorem 4.3 we present method for its calculation.

Theorem 4.3 Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{H}^2$, fuzzy relation $contain$ be *fuzzy contain*. Then a membership function of the fuzzy relation *fuzzy colinearity* is determined according to the following formula

$$\mu_{coli}(\tilde{A}, \tilde{B}, \tilde{C}) = \max\left(\mu_{contain}(\tilde{A}, \widetilde{BC}), \mu_{contain}(\tilde{B}, \widetilde{AC}), \mu_{contain}(\tilde{C}, \widetilde{AB})\right).$$

6. Conclusion

This report presents a mathematical model of the imprecise basic plane geometric object (fuzzy line, fuzzy triangle, fuzzy circle), proven, theirs main properties, introduced basic distance functions and introduced the basic imprecise spatial relations (coincidence, contain and collinear).

This model of the imprecise line object is based on the model of fuzzy imprecise point proposed in [1] as the union of linear combination of two fuzzy points. Consequence is that fuzzy line can be represented by only two fuzzy points, which is simple, yet descriptive extension of precise ideal line. Imprecise spatial relations are based on fuzzy relations between fuzzy points and fuzzy lines .

The proposed models of imprecise line objects are primarily intended for GIS (imprecise spatial object modeling), but also apply to various other domains such as image analysis (imprecise feature extraction), robotics (environment models), etc.

References

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